

Passage of radiation through wormholes

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We investigate numerically the process of the passage of a radiation pulse through a wormhole and the subsequent evolution of the wormhole that is caused by the gravitational action of this pulse. The initial static wormhole is modeled by the spherically symmetrical Armendariz-Picon solution with zero mass. The radiation pulses are modeled by spherically symmetrical shells of self-gravitating massless scalar fields. We demonstrate that the compact signal propagates through the wormhole and investigate the dynamics of the fields in this process for both cases: collapse of the wormhole into the black hole and for the expanding wormhole.

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I. INTRODUCTION

One of the most interesting features of the theory of general relativity is the possible existence of spacetimes with wormholes [1, 2, 3, 4, 5, 6]. The wormholes are topological tunnels which connect different asymptotically flat regions of our Universe or different universes in the model of Multiverse [7].

Recently the hypothesis that some known astrophysical objects (e.g. the active nuclei of some galaxies) could be entrances to wormholes was considered by Kardashev, Novikov and Shatskiy [8, 9] and by Shatskiy [10]. If some of these wormholes are traversable, then radiation and information, at least in principle, can pass between two regions of the Universe or from one universe to another universe in the model of the Multiverse. Furthermore, if a wormhole exists, in principle it is possible to transform it into a time machine (space-time with closed time-like curves) [11, 12]. Therefore, it is very important for both the theory and the development of the hypothesis about the real existence of the wormholes to analyze the physics of the passage of radiation through a wormhole. According to the wormhole models of general relativity, in the absence of any special matter, the throat of the wormhole pinches off so quickly that it cannot be traversed even by a test signal moving with the velocity of light [13]. In order to prevent the shrinking of a wormhole and to make it traversable, it is necessary to thread its throat with so-called "exotic matter", which is matter that violates the averaged null energy condition (see [5, 14, 15]). Many questions associated with the existence of wormholes and with exotic matter remain unsolved, e.g. its stability against various processes, and different views have been expressed in the literature (see for example [5, 13, 16, 17]). We will not discuss this here. Different types of wormholes may exist depending on the type of exotic matter in their throats (see [9, 18, 19]). For example, it could be a "magnetic exotic matter" in which the main component is a strong, ordered, magnetic field plus a small amount of a "true

exotic matter" (see [9]). Another type of exotic matter is a "scalar exotic matter" in the form of a scalar field with a negative energy density (see [18]). Finally, it can be in the form of a mixed "magnetic-negative dust exotic matter" type in which it is a mixture of an ordered magnetic field and dust (matter with zero pressure) with negative matter density (see [19]). The physical properties of different types of wormholes are different (see for example [19]). The possible consequences of the passage of radiation through a wormhole with a "scalar exotic matter" was considered, using an analytical approach, by Doroshkevich, Kardashev, D. Novikov and I. Novikov in [20], where a toy model was used to accept some very artificial hypotheses about the final state of the wormhole. The results obtained in [20] allowed the authors to reach some important conclusions, but of course without numerical computations, the real dynamics of the processes of radiation propagation through a wormhole could not be considered. Other important analytical analyses of the existence and evolution of the wormholes and their possible transformation into black holes (BH) was seen in [21, 22, 23, 24, 25, 26, 27, 28]. Other important results are also seen in the papers [29, 30, 31].

The new period of the investigation of the wormholes began from the seminal papers [32, 33], where the wormhole dynamics were analyzed using numerical simulations. Paper [32] is devoted mainly to the evolution of an initial static wormhole maintained by the scalar exotic matter under the gravitational action of the scalar radiation pulse passing the wormhole from one asymptotically flat region to other one. In paper [33] the authors analyze the nonlinear evolution of the wormhole perturbed by the scalar field.

The goal of this paper is to continue the line of the investigation of these two works. We focus on the dynamics of the scalar fields during the nonlinear stages of the wormhole evolution under the gravitational action of the ingoing pulse of the scalar radiation (both exotic and usual). These dynamics can explain some results of the papers [32, 33].

This paper is organized as follows:

In section II we present the initial model of the wormhole. In section III the field equations are written out. The initial value problem is discussed in section IV. In section V we present the numerical solution of the field equation which describe the passage of the signal through the wormhole and its subsequent evolution with the formation of a BH. In section VI we analyze the case of a strong signal. The passage of the signal of the exotic scalar field is considered in section VII. Finally we summarize our conclusions in section VIII. Mathematical details and the details of the numerical code are given in appendices A-C.

II. THE MODEL

We study, numerically, the evolution of a simple spherical model of a wormhole maintained by an exotic scalar field and perturbed by an in-falling massless, self-gravitating scalar field. This initially in-falling scalar field is a pulse in the form of a spherical layer with some width and amplitude. It imitates a pulse of radiation directed into the wormhole and propagating with velocity of light. We do not impose that the in-falling scalar field is weak. We will consider two cases: when this in-falling field has a positive energy density (imitating real radiation), and when it has a negative energy density (imitating exotic radiation). As an initial static wormhole, let us consider the special case of the Armendariz-Picon solution of the Einstein equations for the spherical static state with the effective mass equal to zero [6, 18]. Another name of this wormhole is Moris-Thorne's (MT) wormhole. The corresponding line element for such a wormhole can be written as follows:

$$ds^{2(MT)} = -d\tau^2 + dR^2 + r^2 d\Omega^2, \quad r^{2(MT)} = Q^2 + R^2, \quad (1)$$

where R is running from $-\infty$ to $+\infty$, Q is a characteristic of the strength of the exotic supporting field Ψ and $d\Omega^2 = d\theta^2 + \sin^2 \theta d\varphi^2$ is the line element on the unit two-sphere. The general equations will be given below. Here we give the non-zero components of the energy-momentum tensor of the Ψ -field for the solution (1):

$$T_\tau^{(MT)} = \frac{Q^2}{8\pi r^4} \quad (2a)$$

$$T_R^{(MT)} = -\frac{Q^2}{8\pi r^4} \quad (2b)$$

$$T_\theta^{(MT)} = T_\varphi^{(MT)} = \frac{Q^2}{8\pi r^4} \quad (2c)$$

The narrowest part of the wormhole (1) is at $R = 0$. This throat corresponds to $r = Q$. The physical analysis of the metric (1) is given in [20]. Here we emphasize that for the metric (1), in this reference frame, there is no gravitational acceleration at any point in the three-dimensional

space, but the wormhole (two asymptotically flat three-dimensional regions joined by a three-dimensional tunnel) nevertheless exists due to the distribution (2) of the exotic scalar field. Of course there are not any apparent and event horizons in such an object and test signals can pass through the tunnel in both directions.

The initial motivations of our choice of the MT wormhole as an initial static wormhole were the following.

(1) The corresponding solutions (1)-(2) of the Einstein equations are very simple and elegant.

(2) It was declared in [18] that the MT wormhole is stable.

However the analytical [34] and numerical [33] analysis (see also Appendix A) demonstrates that the solutions (1)-(2) are unstable against small spherical perturbations. Our numerical experiments confirms this conclusion.

In the light of this fact, the analysis of the possibility of the passage of the signals through the MT wormhole and their distortions have a special interest.

III. FIELD EQUATIONS

For the numerical analysis we will use the double null coordinates. The general line element in these coordinates can be written as:

$$ds^2 = -2e^{2\sigma(u,v)} du dv + r^2(u,v) d\Omega^2, \quad (3)$$

where $\sigma(u,v)$ and $r(u,v)$ are functions of the null coordinates u and v (in- and out-going respectively). The non-zero components of the Einstein tensor are:

$$G_{uu} = \frac{4r_{,u}\sigma_{,u} - 2r_{,uu}}{r} \quad (4a)$$

$$G_{vv} = \frac{4r_{,v}\sigma_{,v} - 2r_{,vv}}{r} \quad (4b)$$

$$G_{uv} = \frac{e^{2\sigma} + 2r_{,u}r_{,v} + 2rr_{,uv}}{r^2} \quad (4c)$$

$$G_{\theta\theta} = -2e^{-2\sigma}r(r_{,uv} + r\sigma_{,uv}) \quad (4d)$$

$$G_{\varphi\varphi} = -2e^{-2\sigma}r \sin^2 \theta (r_{,uv} + r\sigma_{,uv}) \quad (4e)$$

The energy-momentum tensor can be written as a sum of contributions from the exotic scalar field Ψ and the ordinary scalar field Φ with the positive energy density: $T_{\mu\nu} = T_{\mu\nu}^\Psi + T_{\mu\nu}^\Phi$:

$$T_{\mu\nu}^\Psi = \frac{-1}{4\pi} \begin{pmatrix} \Psi_{,u}^2 & 0 & 0 & 0 \\ 0 & \Psi_{,v}^2 & 0 & 0 \\ 0 & 0 & r^2 e^{-2\sigma} \Psi_{,u} \Psi_{,v} & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \theta e^{-2\sigma} \Psi_{,u} \Psi_{,v} \end{pmatrix} \quad (5)$$

$$T_{\mu\nu}^\Phi = \frac{+1}{4\pi} \begin{pmatrix} \Phi_{,u}^2 & 0 & 0 & 0 \\ 0 & \Phi_{,v}^2 & 0 & 0 \\ 0 & 0 & r^2 e^{-2\sigma} \Phi_{,u} \Phi_{,v} & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \theta e^{-2\sigma} \Phi_{,u} \Phi_{,v} \end{pmatrix} \quad (6)$$

The $u-u$, $v-v$, $u-v$ and $\theta-\theta$ components of the Einstein equations respectively are (with $c = 1$, $G = 1$):

$$r_{,uu} - 2r_{,u}\sigma_{,u} - r(\Psi_{,u})^2 + r(\Phi_{,u})^2 = 0 \quad (7)$$

$$r_{,vv} - 2r_{,v}\sigma_{,v} - r(\Psi_{,v})^2 + r(\Phi_{,v})^2 = 0 \quad (8)$$

$$r_{,uv} + \frac{r_{,u}r_{,v}}{r} + \frac{e^{2\sigma}}{2r} = 0 \quad (9)$$

$$\sigma_{,uv} - \frac{r_{,v}r_{,u}}{r^2} - \frac{e^{2\sigma}}{2r^2} - \Psi_{,u}\Psi_{,v} + \Phi_{,u}\Phi_{,v} = 0 \quad (10)$$

The scalar fields satisfy the Gordon-Klein equation $\nabla^\mu \nabla_\mu \Psi = 0$ and $\nabla^\mu \nabla_\mu \Phi = 0$, which in the metric (3) become:

$$\Psi_{,uv} + \frac{1}{r}(r_{,v}\Psi_{,u} + r_{,u}\Psi_{,v}) = 0 \quad (11)$$

$$\Phi_{,uv} + \frac{1}{r}(r_{,v}\Phi_{,u} + r_{,u}\Phi_{,v}) = 0 \quad (12)$$

Equations (9)-(12) are evolution equations which are supplemented by the two constraint equations (7) and (8). It is noted that none of these equations depends directly on the scalar fields Ψ and Φ , but only on their derivatives, i.e. the derivative of the scalar field is a physical quantity, while the absolute value of the scalar field itself is not. Specifically we note the $T_{uu}^\Psi = -(\Psi_{,u})^2/(4\pi)$ and $T_{vv}^\Psi = -(\Psi_{,v})^2/(4\pi)$ components of the energy-momentum tensor (5) and $T_{uu}^\Phi = (\Phi_{,u})^2/(4\pi)$ and $T_{vv}^\Phi = (\Phi_{,v})^2/(4\pi)$ components of the energy-momentum tensor (6), which are part of the constraint equations. Physically T_{uu} and T_{vv} represents the flux of the scalar field through a surface of constant v and u respectively. These fluxes will play an important role in our interpretation of the numerical results in sections V, VI and VII.

IV. INITIAL VALUE PROBLEM

We wish to numerically evolve the unknown functions $r(u, v)$, $\sigma(u, v)$, $\Phi(u, v)$ and $\Psi(u, v)$ throughout some computational domain. We do this by following the approach of [35, 36, 37, 38, 39, 40, 41, 42, 43] to numerically integrate the four evolution equations (9)-(12). These equations form a well-posed initial value problem in which we can specify initial values of the unknowns on two initial null segments, namely an ingoing ($v = v_0 = \text{constant}$) and an outgoing ($u = u_0 = \text{constant}$) segment. We impose the constraint equations (7) and (8) on the initial segments. Consistency of the evolving fields with the constraint equations is then ensured via the contracted Bianchi identities [37], but we use the constraint equations throughout the domain of integration to check the accuracy of the numerical simulation.

Our choice of the initial values corresponds to the following physical situation; There is an MT-wormhole and at some distance from one of the entrances, at the initial

moment there is a rather narrow spherical layer of an in-falling scalar field. There is not any radiation coming to the wormhole from the side of the other entrance. As mentioned, we specify the initial values on two initial segments, namely $u = \text{constant}$ and $v = \text{constant}$. In the next section V we consider the case where the in-falling layer consists of the Φ -field. Later in the section VII we consider the case of the in-falling layer consisting of the Ψ -field.

For the case of the layer from the Φ -field, the initial condition could be specified as follows:

First, let us consider the static MT solution [18]:

$$\Psi_{,R}^{(MT)} = \frac{Q}{r^2} = \frac{Q}{Q^2 + R^2} \Rightarrow \Psi^{(MT)} = \arctan\left(\frac{R}{Q}\right) \quad (13)$$

In $u-v$ coordinates we can write the MT solution as:

$$r(u, v) = \sqrt{Q^2 + \frac{1}{4}(v-u)^2} \quad (14)$$

$$e^{-2\sigma^{(MT)}} = 2 \quad (15)$$

$$\Psi^{(MT)} = \arctan\left(\frac{v-u}{2Q}\right) \quad (16)$$

$$\Phi^{(MT)} = 0 \quad (17)$$

Equations (14) - (16) completely specifies the MT-wormhole in $u-v$ coordinates in all of our computational domain including the initial surfaces $u = \text{constant}$ and $v = \text{constant}$.

Now, let us consider the more general case with non-trivial Φ and Ψ scalar fields. In sections V and VI we consider the case of a non-zero Φ field and in section VII we consider the case of a perturbed MT Ψ field on the initial outgoing $u = \text{constant}$ surface. In general we are free to choose the Φ and Ψ fields (and their derivatives) on the initial surfaces in any way we wish. Our choices for the Φ and Ψ -fields are described in sections V - VII. We are also free to choose $r(u, v)$ on the initial surfaces, this merely expresses the gauge freedom associated with the transformation $u \rightarrow \tilde{u}(u)$, $v \rightarrow \tilde{v}(v)$ (the line element (3) and the equations (7)-(12) are invariant to such a transformation). We continue to use eq. (14) on the initial surface as our choice of gauge. Hence, the only variable left for us to specify on the initial surfaces is σ . This can easily be found by integrating the constraint equations eq. (7) and (8), which ensures that the constraint equations are satisfied on the initial hypersurfaces. Specifically on the outgoing $u = u_0$ hypersurface it is found by the integral:

$$\sigma(u_0, v) = \ln\left(\frac{1}{\sqrt{2}}\right) + \int_0^v \frac{r_{,vv} - r(\Psi_{,v})^2 + r(\Phi_{,v})^2}{2r_{,v}} dv \quad (18)$$

The initial conditions on the initial ingoing hypersurface is set equal to the MT solution for all simulations. Hence, by specifying a distribution of the scalar fields Φ and Ψ on the initial null segments, choosing a gauge and

charge parameter " Q " we can specify complete initial conditions on the initial null segments. Using a numerical code (described in appendix B), we can then use the evolution equations, eqs. (9)-(12) to evolve the unknown functions throughout the computational domain.

V. PHYSICS OF THE PASSAGE OF THE Φ -FIELD PULSE THROUGH THE MT-WORMHOLE

We investigate the full nonlinear processes arising in the case of propagation of a compact pulse of the scalar field through an MT-wormhole, using results of our numerical simulations. Our numerical code is described in appendix B and tested in appendix C. In this section we analyze the passage of the Φ -field pulse. In section VII we investigate the passage of the Ψ -field pulse. Throughout this paper, the constant Q (see eq. (1)), prior to influence from scalar pulses, has initial value of $Q = 1$. The computational domain for all results throughout this paper is $v = [8, 28]$ and $u = [0, 20]$. The flux of the scalar Φ -field into the wormhole is specified along initial $u = u_0 = 0$ outside the wormhole in the following way:

$$\Phi_{,v}(u_0, v) = A^\Phi \sin^2 \left(\pi \frac{v - v_1}{v_2 - v_1} \right) \quad (19)$$

where v_1 and v_2 marks the beginning and end of the ingoing scalar pulse, respectively, and A^Φ measures the amplitude of the pulse. Before and after the pulse, at $v < v_1$ and $v > v_2$ the flux through $u = u_0$ is set equal to zero, i.e. $\Phi_{,v}(u_0, v) = 0$. The flux of the scalar Φ -field through initial segment $v = v_0$ is set equal to zero: $\Phi_{,u}(u, v_0) = 0$. The expression (19) can readily be integrated to give:

$$\Phi(u_0, v) = \frac{A^\Phi}{4\pi} \left(2\pi(v - v_1) - (v_2 - v_1) \sin \left(2\pi \frac{v - v_1}{v_2 - v_1} \right) \right) \quad (20)$$

Note that we formulate the initial condition directly for the flux $T_{vv} = (\Phi_{,v})^2 / (4\pi)$ of the scalar field through the surface $u = u_0$, rather than for Φ itself since T_{vv} has the direct physical meaning. Also remember from section IV that once the flux through the two initial surfaces has been chosen, all other initial conditions are determined by the model. In the examples of the results of our computations we specify $v_1 = 9$ and $v_2 = 11$. In our computations we vary the amplitude A^Φ of the pulse.

In Fig. 1 is seen the static case of the MT metric (1) in $u - v$ coordinates, without any perturbations of the scalar fields. The throat $r = 1$ corresponds to the diagonal $u = v$.

In Fig. 2 one can see a typical example of the evolution of the wormhole due to the action of the in-falling Φ scalar field. Initial values for this Φ -field are $v_1 = 9$, $v_2 = 11$, $A^\Phi = 0.01$. Fig. 2(a) shows the evolution of the lines $r = \text{const}$. As is seen in Fig. 2(a), the evolution corresponds to contraction of the throat down to $r = 0$, and contraction of the whole wormhole. The real singularity

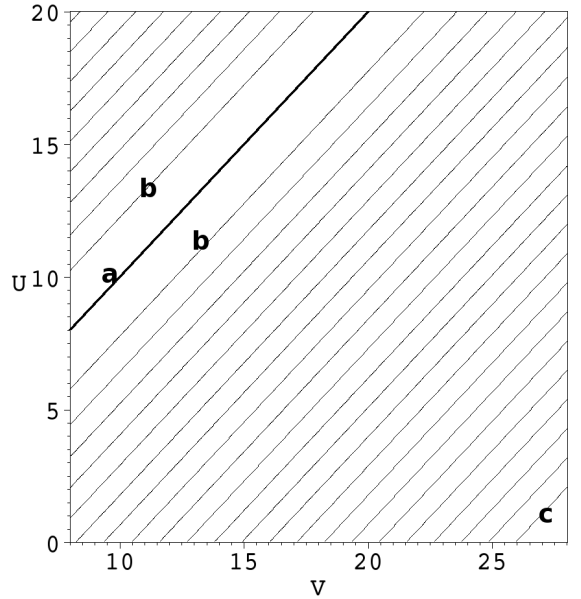
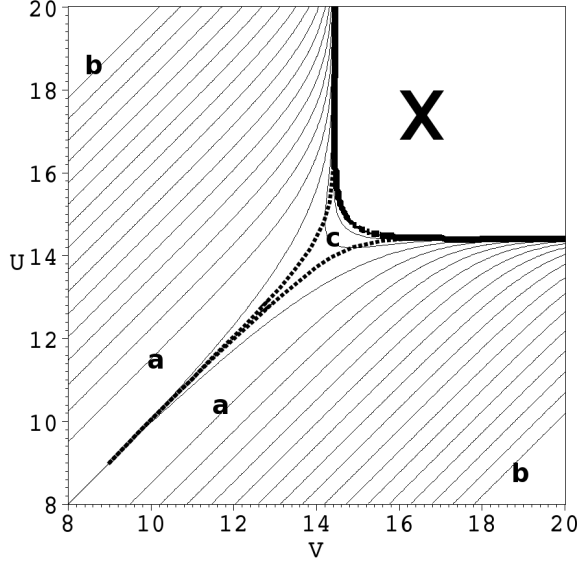


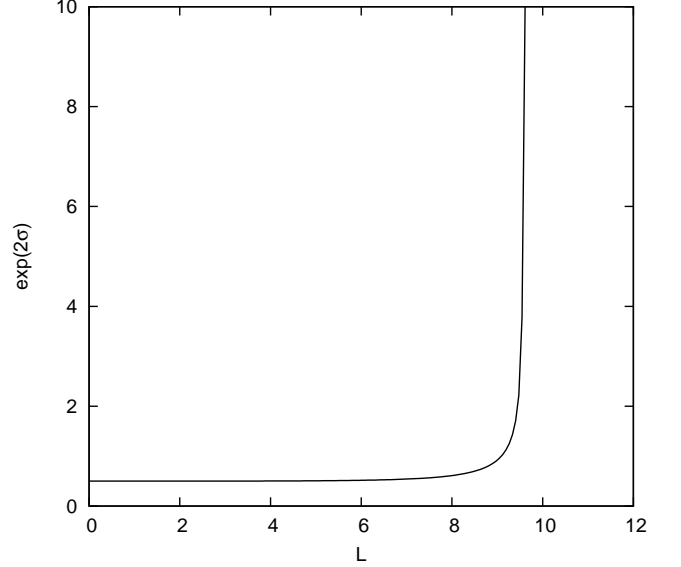
FIG. 1: Static MT-wormhole. Plot shows contour lines of constant r . Thick diagonal line "a" marks the throat of the wormhole (corresponding to the diagonal $u = v$) with $r = 1$. The thin lines mark lines of constant r increasing outwards from the throat of the wormhole with a constant spacing between lines of $\Delta r = 0.5$ (i.e. line "b" marks $r = 1.5$ up to line "c" at $r = 12$).

of the space-time arises at $r = 0$. It is seen from the fact that when we come to $r = 0$, the metric coefficient $e^{2\sigma}$ in formula (3) tends to infinity (see Fig. 2(b)). Also apparent and event horizons arise. The apparent horizon corresponds to events where world lines $r = \text{const}$ are horizontal or vertical (see [44] page 85). All space-time is divided into R - and T -regions (see [13, 45, 46, 47]). In the R -regions (lines "a" and "b"), the lines $r = \text{const}$ are time-like and signals can go both to higher and smaller r . In the T -region (between the two apparent horizons) the lines $r = \text{const}$ are space-like and a signal can propagate to smaller r only. The structure of the apparent horizon depends on the value of the flux of the energy through it. We will analyze it later (see section VI).

Now we consider the evolution of the Φ - and Ψ -fields. Fig. 3(a) represents the evolution of the flux T_{vv}^Φ of the scalar Φ -energy into the wormhole. Fig. 3(b) shows the evolution of the T_{uu}^Φ flux which arises as a result of T_{vv}^Φ being scattered by the space-time curvature. Fig. 4(a) represents the evolution of the difference of the $T_{vv}^\Psi - T_{uu}^\Psi$ of the fluxes of the Ψ -field in and out of the wormhole. The reason why for the Ψ -field we show the difference of the in and out fluxes rather than the T_{vv}^Ψ and T_{uu}^Ψ -fluxes separately, is the fact that even for the static MT-solution (1), (2), the values of T_{vv}^Ψ and T_{uu}^Ψ are not equal to zero (but equal to each other). So only the difference between them is important for the dynamics. Finally Fig. 4(b) -



(a) Lines of constant r , ranging from $r = 0$ to $r = 5$ with a spacing between lines of $\Delta r = 0.25$ (line "a" marks $r = 1.25$, line "b" marks $r = 5$ and line "c" marks $r = 0.75$). The thick dotted lines marks the position of the apparent horizons. The thick fully drawn line marks the position of the singularity $r = 0$, hence the region marked "X" is outside of the computational domain.



(b) Plot of the metric coefficient $e^{2\sigma(L)}$ as a function of distance parameter $L = \sqrt{(v - v_0)^2 + (u - u_0)^2}$ measuring the distance along the wormhole throat at $u = v$ (with $L = 0$ corresponding to where the Φ -pulse first reaches the wormhole throat, i.e. $v_0 = u_0 = 9$).

FIG. 2: Results of simulation with a non-zero Φ -field modeled after eq. (20) with $v_1 = 9, v_2 = 11$ and amplitude $A^\Phi = 0.01$.

4(d) shows the comparison of all three fluxes at different slices:

$$u = 3, 14, 20.$$

In Fig. 3(a) and 3(b) the initial pulse (between $9.0 < v < 11.0$) and subsequent tails with resonances are clearly seen. In different regions, T_{uu}^Φ and T_{vv}^Φ are converted into one another due to curvature scattering and resonances. But at later u (see Fig. 4(c), 4(d)) and for $v > 11$ the flux $T_{vv}^\Psi - T_{uu}^\Psi$ of the Ψ -field dominates absolutely (in modulus). Thus for these regions this flux of the Ψ -field determines the dynamics of the wormhole.

From the Fig. 2-4 the general picture of the evolution of the wormhole under the action of the passage of the compact Φ -field signal looks like the following. At the beginning, a rather weak signal produces small perturbations of the static wormhole. These perturbations trigger the evolution of the Ψ -field that maintains the wormhole. The fluxes of the Ψ -field (in- and out-fluxes) determine the subsequent evolution of the wormhole and lead to its collapse. At the beginning of this process the fluxes of the Ψ -field are directed to the throat from both sides of the wormhole (regions A and B in fig. 4(a)). This flux to the throat of the Ψ -field with the negative energy density may lead to the formation of the negative local effective mass. Remember that before the beginning of the process the local effective mass was equal to

zero everywhere (no any gravitational forces). Just after the passage of the perturbing compact ingoing signal of the Φ -field, the effective mass become positive by the mass of the ingoing Φ -shell. In the process of the collapse the fluxes directed in the opposite directions (out of the throat) arise (regions C and D in fig. 4(a)). These fluxes partly propagate out of the openings of the wormhole and partly propagate inside of the forming BH. The fluxes of the Ψ -field going out of the wormhole carry away negative energy providing origin of the positive mass of the forming BH. That is emphasized in papers [32, 33]. Everything looks like a collapse and an explosion of the Ψ -field.

A very important fact is that the compact signal of the Φ -field propagates through the wormhole from one asymptotically flat region into another asymptotically flat region before the beginning of the collapse, this is especially clear from fig. 3(a). Only a small part of the Φ -signal is scattered back and into the arising BH. After the BH formation the propagation of a signal from one asymptotically flat space into another one is impossible in any direction.

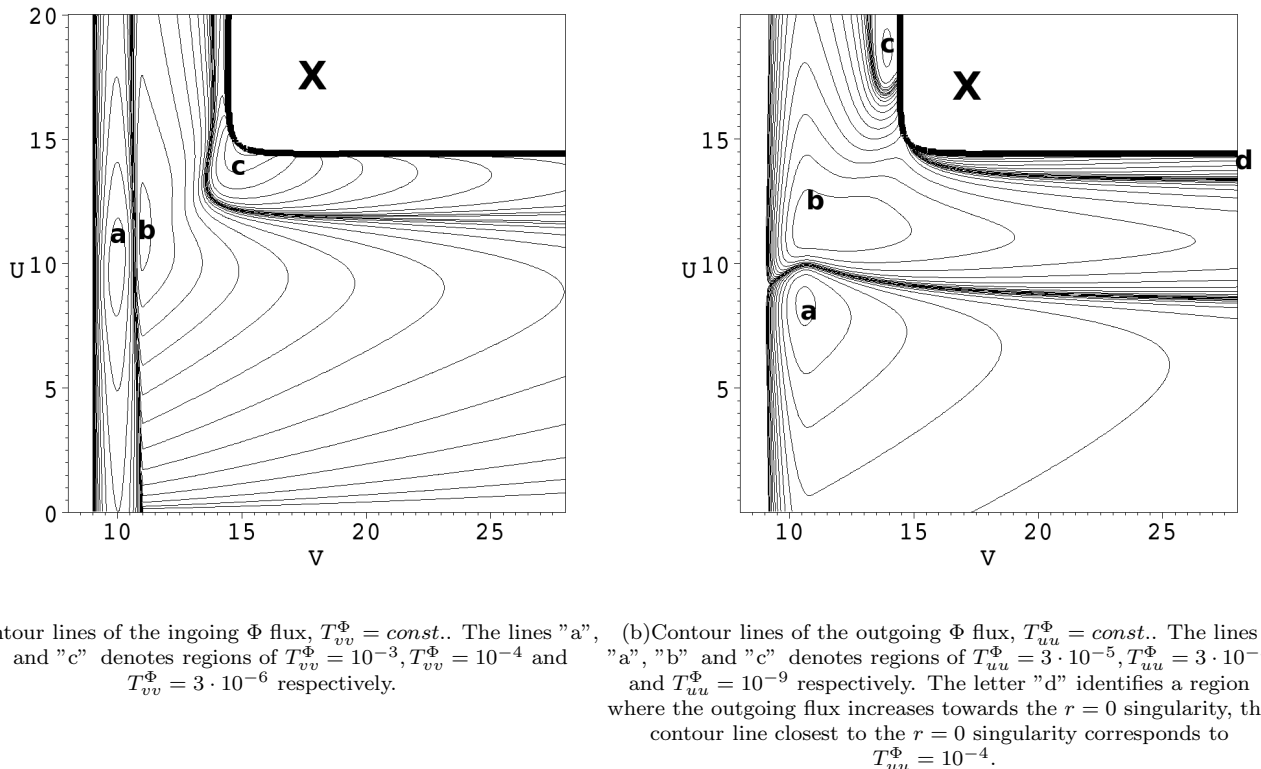


FIG. 3: Contour lines of the in- and out-going Φ fluxes for the simulation with $v_1 = 9, v_2 = 11$ and amplitude $A^\Phi = 0.01$. In both plots lines are logarithmically spaced with a factor $10^{\frac{1}{2}}$ between the lines. The thick solid line marks the location of the $r = 0$ singularity and the region "X" marks the area outside of the computational domain.

VI. STRONGER Φ -SIGNAL, HORIZONS AND RATE OF THE COLLAPSE

Now let us consider the case of an essentially stronger infalling Φ -signal with $A^\Phi = 0.5$ (see Fig. 5(a)-5(d)).

In this case, the collapse of the wormhole arises much faster. The essential part of the initial Φ -signal comes into the arising BH, this can be seen from figure 5(b). The picture of the process is very asymmetrical.

The asymmetry is clearly seen at the picture of the apparent horizon (figure 5(a)). In general, with any strength of the compact Φ -signal, the apparent horizon arises at the throat just at the moment of the passage of the Φ -signal (see Fig. 2(a)). At the beginning the dynamics of the wormhole is rather slow (if the Φ -signal is weak enough) and two branches of the apparent horizon are very close from each other.

When the fast collapse begins these branches are going in different directions.

In the case of a strong initial pulse of the Φ -field (Fig. 5(a)) the shape of the branch of the apparent horizon from the side of the coming Φ -field signal is quite different from the one at the opposite side.

The analogous change of the apparent horizon under the action of the incoming signal we observe in the

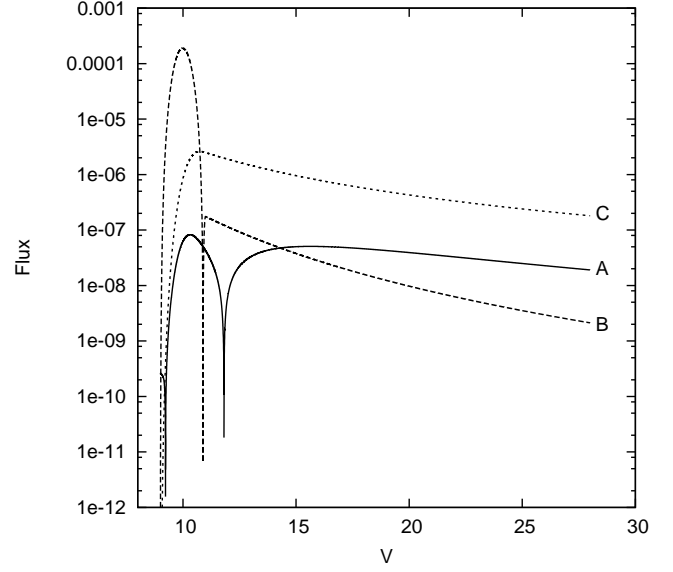
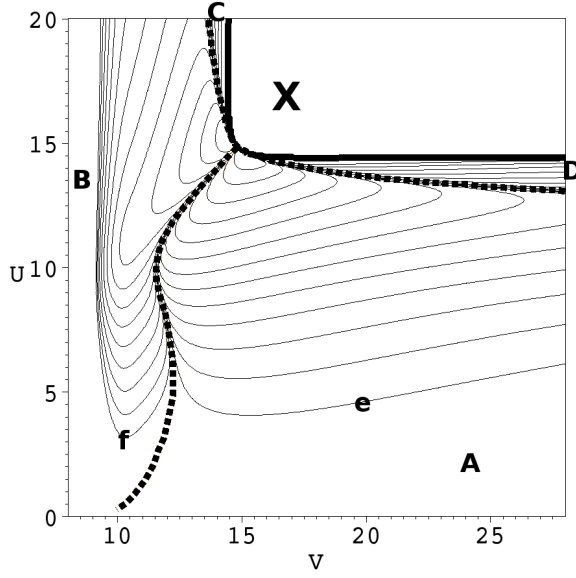
charged BH, see [42].

The picture of the evolution of the Ψ -field in the case of the strong signal (see Fig. 5(d)) is qualitatively similar to the case of weaker signals (Fig. 4(a)).

About the event horizon. From Fig. 2(a) it is clear that events close enough to the singularity $r = 0$ are inside of the BH and thus inside of the event horizon. Indeed, the light signal from these events (vertical and horizontal lines) will come to the singularity and cannot go to infinity. At Fig. 2(a) event horizon practically coincide with vertical and horizontal asymptotes to the singularity $r = 0$. The analogous region inside the BH can be found at the Fig. 5(a).

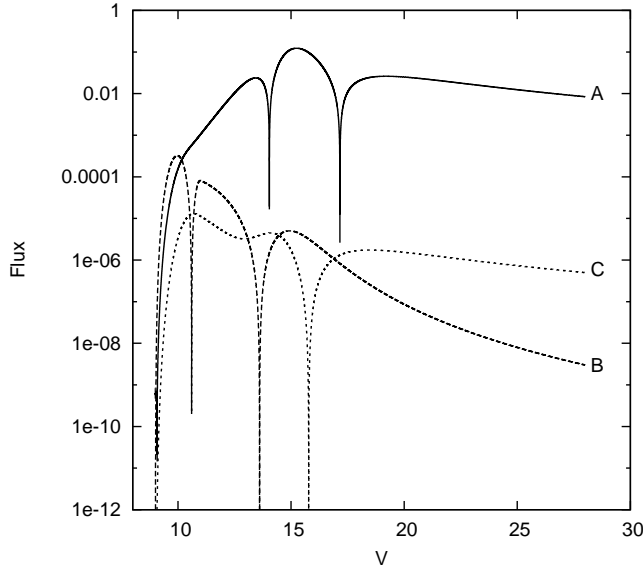
Now let us consider the process of the collapse of the throat. In Fig. 6(a) one can see this process for different amplitudes A^Φ of the ingoing signal of the Φ -field. It is seen that for small A^Φ the wormhole practically does not responds to the Φ -signal during long periods and after that collapses very fast. It looks like a very nonlinear process.

Fig. 6(b) represents the dependence of the moment of collapse of the throat on the amplitude A^Φ of the ingoing Φ -signal. It is seen that for small A^Φ the period between the passage of the signal and the moment of the collapse can be very long.

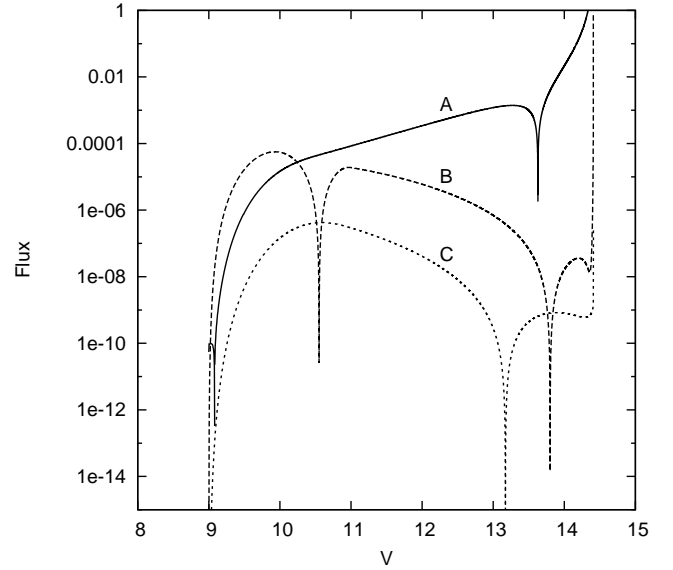


(a) Contour lines of the resulting Ψ -flux $= T_{vv}^{\Psi} - T_{uu}^{\Psi}$ with a factor of $10^{\frac{1}{2}}$ between lines. The lines "e" and "f" denotes $T_{vv}^{\Psi} - T_{uu}^{\Psi} = +10^{-7}$ and $T_{vv}^{\Psi} - T_{uu}^{\Psi} = -10^{-7}$ respectively, with the resulting absolute flux increasing towards the $r = 0$ singularity. The regions marked "A" and "C" marks $T_{vv}^{\Psi} - T_{uu}^{\Psi} > 0$, "B" and "D" marks regions with $T_{vv}^{\Psi} - T_{uu}^{\Psi} < 0$. The thick dotted line marks the contour line $T_{vv}^{\Psi} - T_{uu}^{\Psi} = 0$ and the fully drawn thick line marks the $r = 0$ singularity as in previous plots as well as the region "X" which marks the area outside of the computational domain.

(b) Comparison of the Φ and Ψ fluxes along the line $u = 3$. Line "A" marks $|T_{vv}^{\Psi} - T_{uu}^{\Psi}|$, line "B" marks T_{vv}^{Φ} and line "C" marks T_{uu}^{Φ} .

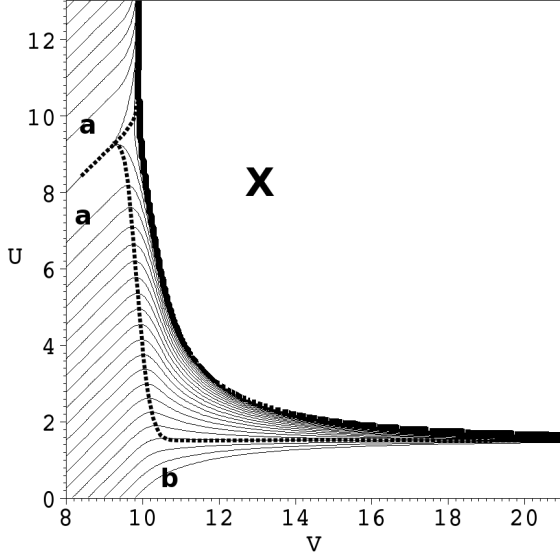


(c) Comparison of the Φ and Ψ fluxes along the line $u = 14$. Line "A" marks $|T_{vv}^{\Psi} - T_{uu}^{\Psi}|$, line "B" marks T_{vv}^{Φ} and line "C" marks T_{uu}^{Φ} .

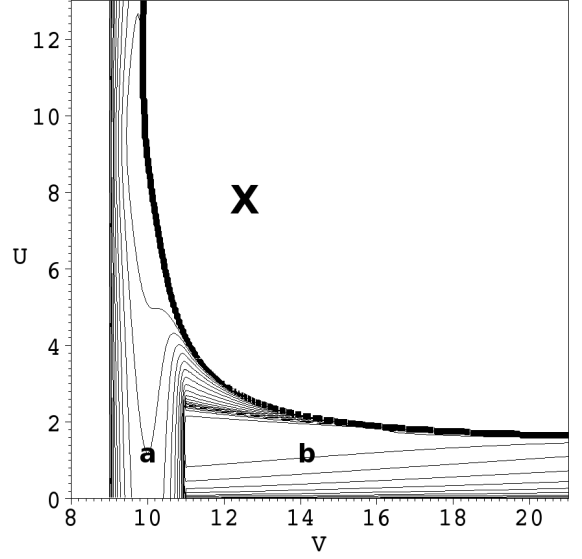


(d) Comparison of the Φ and Ψ fluxes along the line $u = 20$. Line "A" marks $|T_{vv}^{\Psi} - T_{uu}^{\Psi}|$, line "B" marks T_{vv}^{Φ} and line "C" marks T_{uu}^{Φ} .

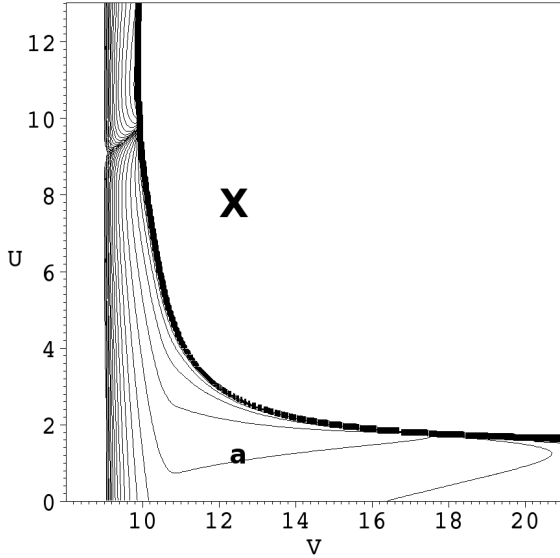
FIG. 4: Results of the simulation with $v_1 = 9$, $v_2 = 11$ and amplitude $A^{\Phi} = 0.01$.



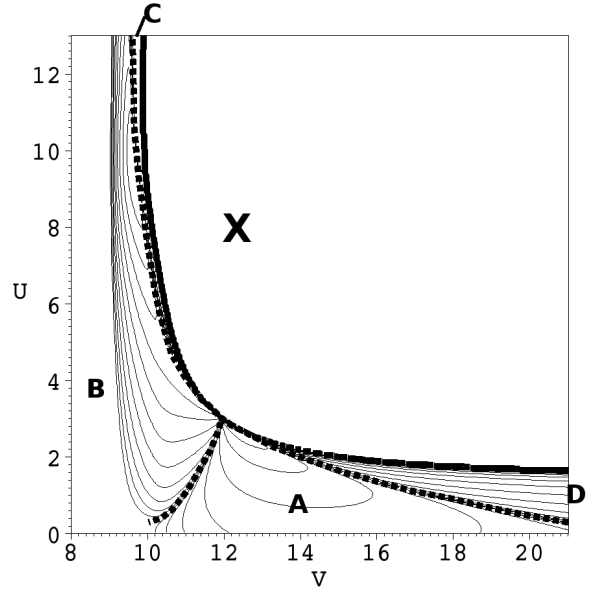
(a) Lines of constant r , from $r = 0$ to $r = 5$ with $\Delta r = 0.2$ (lines "a" marks $r = 1.2$ and line "b" marks $r = 5$). Thick dotted line marks the position of the apparent horizons.



(b) Lines of constant T_{vv}^Φ . Lines are logarithmically spaced with a factor $10^{\frac{1}{2}}$ between lines, line "a" marks $T_{vv}^\Phi = 10^{-0.5}$ and line "b" marks $T_{vv}^\Phi = 10^{-5}$.

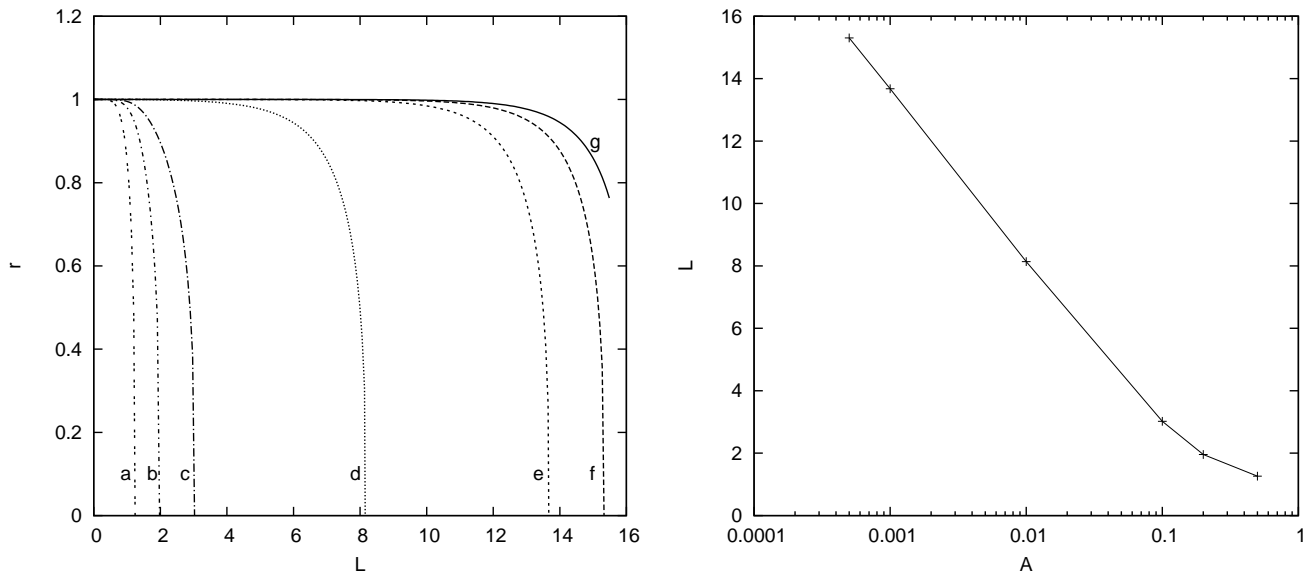


(c) Lines of constant T_{uu}^Φ . Lines are logarithmically spaced with a factor $10^{\frac{1}{2}}$ between lines, line "a" marks $T_{vv}^\Phi = 10^{-2.5}$.



(d) Lines of constant $T_{vv}^\Psi - T_{uu}^\Psi$, Region "A" and "C" marks regions with $T_{vv}^\Psi - T_{uu}^\Psi > 0$, "B" and "D" marks regions with $T_{vv}^\Psi - T_{uu}^\Psi < 0$ and the thick dotted line marks $T_{vv}^\Psi - T_{uu}^\Psi = 0$.

FIG. 5: Results for a simulation with $v_1 = 9, v_2 = 11$ and amplitude $A^\Phi = 0.5$. For all figures, the thick fully drawn line marks the position of the singularity $r = 0$, and the region marked "X" marks area outside of the computational domain.



(a) r as a function of the distance parameter $L = \sqrt{(v-v_0)^2 + (u-u_0)^2}$ with $v_0 = u_0 = 9$. The curves "a", "b", "c", "d", "e", "f" and "g" marks simulations with $A^\Phi = 0.5, 0.2, 0.1, 0.01, 0.001, 0.00035$ and 0.0001 respectively.

(b) Distance parameter $L = \sqrt{(v-v_0)^2 + (u-u_0)^2}$ with $v_0 = u_0 = 9$ measured along $u = v$ from the start of the pulse to the position of the $r = 0$ singularity as a function of scalar field amplitude A^Φ .

FIG. 6: Development of the $r = 0$ singularity as a function of pulse amplitude A^Φ (with identical pulse widths of $v_1 = 9, v_2 = 11$).

VII. PASSAGE OF THE Ψ -FIELD

In this section we briefly consider the passage of the Ψ -field compact pulse through a MT-wormhole. As we told at the end of section II, the MT wormhole is unstable against small spherical perturbations. Here we consider not small perturbations in the form of a compact signal coming from outside. As we will see, the evolution is in agreement with our conclusion about instability of the MT wormhole. Now we put the Φ -field $\equiv 0$ everywhere and model the ingoing Ψ -flux as a perturbation to the background $\Psi^{(MT)}$ field supporting the wormhole:

$$\Psi(u_0, v) = \Psi^{(MT)}(u_0, v) + \bar{\psi}(u_0, v) \quad (21)$$

The form for the perturbation of the Ψ -field into the wormhole is specified almost the same way as in the case of the Φ -field in the section V. But now instead of the formula (19) we should write down

$$\bar{\psi}_{,v}(u_0, v) = A^\Psi \sin^2 \left(\pi \frac{v - v_1}{v_2 - v_1} \right), \quad (22)$$

All other remarks about the initial conditions from section V are correct here also. But now there is one new factor in this case. Namely, this pulse of the Ψ -field is an addition to the background of the Ψ -field of the MT solution. This addition can be taken with the sign plus or minus. This means that A^Ψ can be positive or negative. The evolution depends critically on this sign.

Fig. 7 represent the case of the expression (22) with the sign $A^\Psi > 0$. The wormhole expands (Fig. 7(a)) and the

size of the throat becomes larger and larger. The physical reason for this expansion is clear, the additional amount of the Ψ -field with the negative energy density creates gravitational repulsion. This gives the initial push. The expansion at the late v -parameter looks like an almost "inertial".

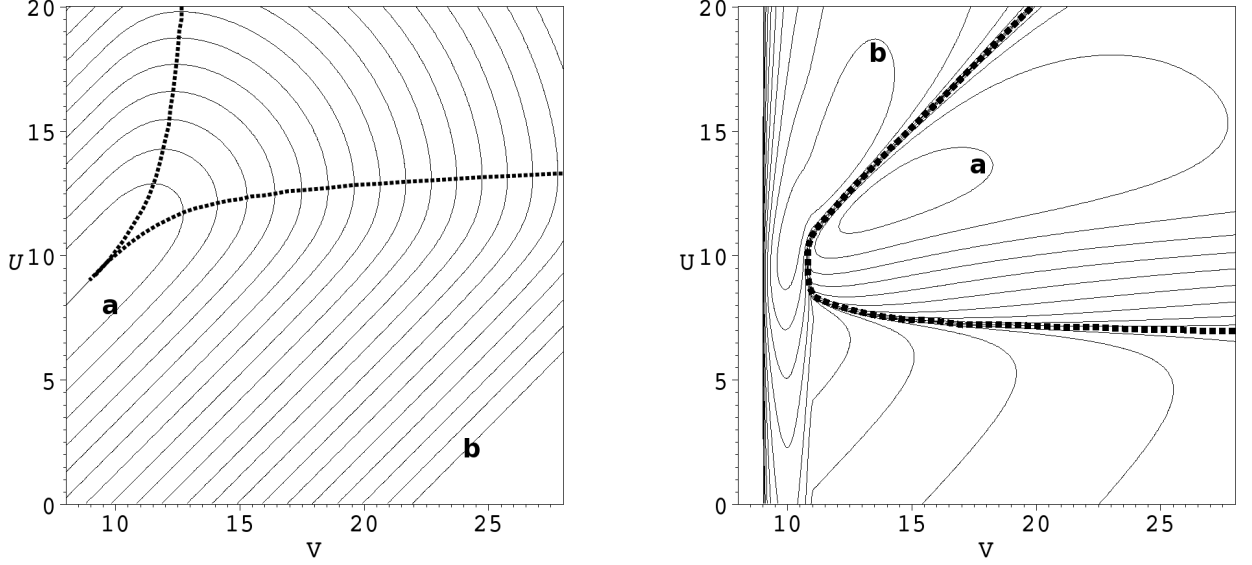
Between two branches of the apparent horizon any test particle or radiation can move to greater and greater r only. This is a so called expanding T_+ region, see [13, 48, 49, 50, 51].

Fig. 7(b) represent the fluxes of the Ψ -field out of the throat to greater r .

The case of the expression (22) with the sign $A^\Psi < 0$ is represented in Fig. 8. Now there is a deficit of the Ψ -field as compared with the static solution (1), (2). It leads to collapse of the wormhole. Qualitatively the process of the collapse is the same as it was described in section V.

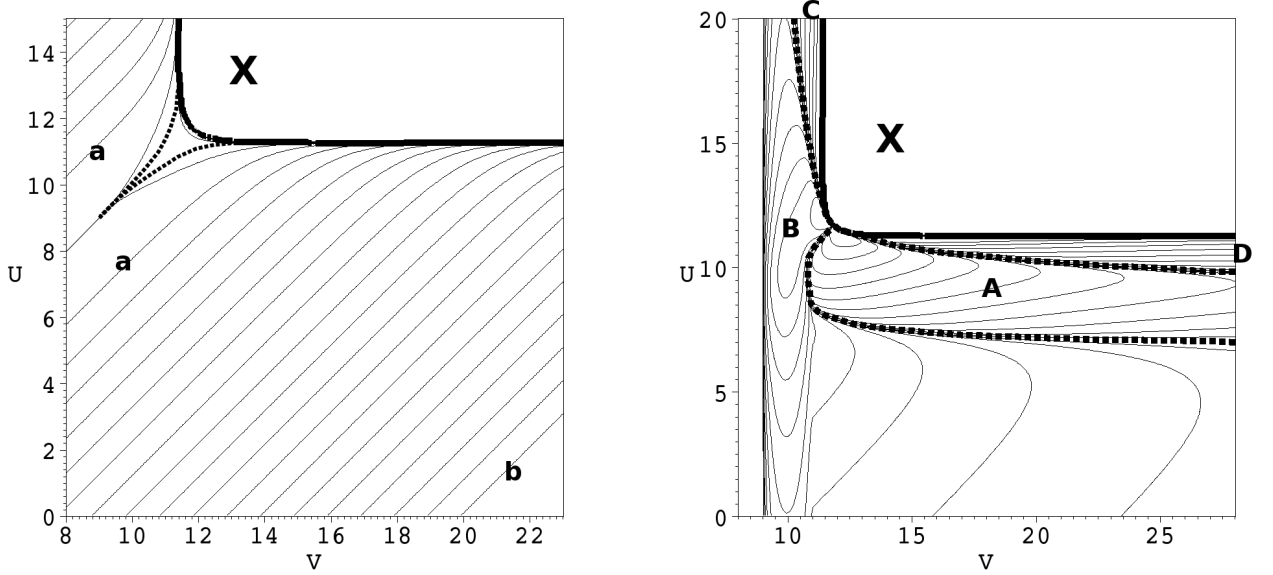
VIII. CONCLUSIONS

In this paper we investigated numerically the process of the passage of the radiation pulse through a wormhole and the subsequent evolution of the wormhole as respond to a gravitational action of this signal. It is the continuation of the investigations started in the papers [32, 33, 34]. In our work we focus mainly on the evolution of the fields Φ and Ψ . We use the MT solution [see (1)-(2)] of the Einstein equations as a model of an initial static wormhole.



(a) Lines of constant r , from $r = 1$ to $r = 5$ with $\Delta r = 0.5$ between lines (line "a" marks $r = 1.5$ and line "b" marks $r = 11$, (b) Lines of constant $T_{vv}^{\Psi} - T_{uu}^{\Psi}$, Region "a" marks region with $T_{vv}^{\Psi} - T_{uu}^{\Psi} < 0$ "b" marks region with $T_{vv}^{\Psi} - T_{uu}^{\Psi} > 0$ and the thick dotted line marks $T_{vv}^{\Psi} - T_{uu}^{\Psi} = 0$.

FIG. 7: Results for a simulation with $v_1 = 9, v_2 = 11$ and $A^{\Psi} = +0.01$ ($\Phi = 0$ everywhere).



(a) Lines of constant r , from $r = 1$ to $r = 10$ with spacing $\Delta r = 0.5$ between lines, lines "a" marks $r = 1.5$, line "b" marks $r = 10$. Thick dotted lines marks positions of apparent horizons. (b) Lines of constant $T_{vv}^{\Psi} - T_{uu}^{\Psi}$, Regions "A" and "C" marks regions with $T_{vv}^{\Psi} - T_{uu}^{\Psi} > 0$, "B" and "D" marks regions with $T_{vv}^{\Psi} - T_{uu}^{\Psi} < 0$ and the thick dotted line marks $T_{vv}^{\Psi} - T_{uu}^{\Psi} = 0$.

FIG. 8: Results for a simulation with $v_1 = 9, v_2 = 11$ and $A^{\Psi} = -0.01$ ($\Phi = 0$ everywhere). As previously, the thick fully drawn line marks the position of $r = 0$ and the region marked "X" indicates the region beyond our computational domain.

The spherically symmetrical radiation pulse was modeled by the self-gravitating, minimally coupled, massless scalar fields Φ and Ψ [see eqs (19)-(22)]. We created and tested a numerical code which is stable and second order accurate. For our computations we used an adaptive mesh refinement approach in both ingoing u and outgoing v directions. We have demonstrated that if an amplitude of the Φ -field, A^Φ , is small enough, then just after the propagation of it through the throat the perturbations of the static wormhole are small. The compact signal of the Φ -field propagates through the wormhole from one asymptotically flat region into another one. The scattering of the signal is small also. But the small perturbations trigger the evolution of the Ψ -field which maintains the wormhole. The fluxes of the Ψ -field (in- and out-fluxes) determine the subsequent evolution of the wormhole and lead to its collapse and a BH arises.

After the BH formation the propagation of a signal from one asymptotically flat space into another one is impossible in any direction.

In the case of the strong enough Φ -signal in-falling into the wormhole, the collapse of the wormhole arises much faster. The essential part of the initial Φ -signal falls into an arising BH.

We analyzed the dependence of the period between the passage of the signal through the throat and the moment of the collapse of the throat in $r = 0$ on the amplitude A^Φ . For smaller and smaller amplitude A^Φ , this period becomes longer and longer.

Note, that the collapse and the BH formation after the passage of the Φ -signal through the wormhole is typical for this special type of the wormholes supported by a Ψ -field. Other types of wormholes may behave quite different. We will investigate such cases in another paper.

Finally we investigated the passage of a compact signal of the Ψ -field through the wormhole and evolution of the Ψ field in this case.

As we emphasized above our work continues the work [32, 33]. We investigate mainly the evolution of the Φ and Ψ fields during the nonlinear evolution of the wormhole. On the other hand we want to emphasize the difference in our approach from the approach in the paper [33]. In [33] the authors formulate the initial condition for perturbations in such way that it corresponds to two fluxes of the scalar fields which propagate in positive and negative directions. The center of the perturbations is at the throat of the wormhole. Our formulation of the initial condition corresponds to a compact signal coming into the wormhole from the right-hand asymptotically flat region. This relates our approach with possible applications to wormhole astrophysics.

IX. ACKNOWLEDGEMENTS

The authors thanks Pavel Ivanov for useful discussions and Sean Hayward, Hisaaki Shinkai, Jose Antonio Gonzalez, Francisco Guzman and Olivier Sarbach, who called

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APPENDIX A: INSTABILITY

For the spherical metric

$$ds^2 = -d\tau^2 + e^\lambda dR^2 + r^2 d\Omega^2, \quad r^2 = e^\eta (R^2 + Q^2). \quad (A1)$$

the Einstein's tensor G_k^i is:

$$G_\tau^\tau = e^{-\lambda} \left[\eta_{,RR} + \frac{\eta_{,R}(3\eta - 2\lambda)_{,R}}{4} + \frac{R(3\eta - \lambda)_{,R}}{Y^2} + \frac{2Q^2 + R^2}{Y^4} \right] - \frac{e^{-\eta}}{Y^2} - \frac{2\lambda_{,\tau}\eta_{,\tau} + \eta_{,\tau}^2}{4} \quad (A2)$$

$$G_R^R = e^{-\lambda} \left[\frac{R^2}{Y^4} + \frac{\eta_{,R}^2}{4} + \frac{R\eta_{,R}}{Y^2} \right] - \frac{e^{-\eta}}{Y^2} - \eta_{,\tau\tau} - \frac{3}{4}\eta_{,\tau}^2 \quad (A3)$$

$$G_\theta^\theta = e^{-\lambda} \left[\frac{Q^2}{Y^4} + \frac{\eta_{,RR} + \eta_{,R}(\eta - \lambda)_{,R}}{4} + \frac{R(2\eta - \lambda)_{,R}}{2Y^2} \right] - \frac{\lambda_{,\tau\tau} + \eta_{,\tau\tau}}{2} - \frac{\lambda_{,\tau}^2 + \eta_{,\tau}^2 + \eta_{,\tau}\lambda_{,\tau}}{4} \quad (A4)$$

$$G_\tau^R = -e^{-\lambda} \left[\eta_{,\tau R} + \frac{(\eta - \lambda)_{,\tau}}{2} \left(\eta_{,R} + \frac{2R}{Y^2} \right) \right] \quad (A5)$$

Here $Y^2 = R^2 + Q^2$.

The equation for the scalar field is

$$\Psi_{;i}^{;i} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^i} \left(\sqrt{-g} g^{ik} \frac{\partial \Psi}{\partial x^k} \right) = 0 \quad (A6)$$

Or:

$$\left[\exp \left(\eta + \frac{\lambda}{2} \right) \Psi_{,\tau} \right]_{,\tau} = \frac{1}{Y^2} \left[\exp \left(\eta - \frac{\lambda}{2} \right) Y^2 \Psi_{,R} \right]_{,R} \quad (A7)$$

The energy-momentum tensor T_k^i is:

$$T_\tau^\tau = \frac{1}{8\pi} [\Psi_{,\tau}^2 + e^{-\lambda} \Psi_{,R}^2] = -T_R^R, \quad (\text{A8})$$

$$T_\theta^\theta = -\frac{1}{8\pi} [\Psi_{,\tau}^2 - e^{-\lambda} \Psi_{,R}^2], \quad T_\tau^R = -\frac{1}{4\pi} e^{-\lambda} \Psi_{,\tau} \Psi_{,R}$$

The static MT solution is

$$\eta = 0, \quad \lambda = 0, \quad \Psi_{,R} = \frac{\pm Q}{Y^2} \quad (\text{A9})$$

$$T_\tau^\tau = -T_R^R = T_\theta^\theta = Q^2 / (8\pi Y^4) \quad (\text{A10})$$

1. Perturbations

Further on we will use dimensionless coordinates $x \equiv R/Q$, $t \equiv c\tau/Q$ and will consider linearized Einstein's equations: $G_k^i = 8\pi T_k^i$. Perturbed components of the energy-momentum tensor are dimensionless. From (A2-A8) for $\eta \ll 1$, $\lambda \ll 1$ we get for perturbations:

$$\delta T_t^t = -\delta T_R^R = \delta T_\theta^\theta = \frac{\Psi_{,x}}{8\pi} [2\bar{\psi}_{,x} - \lambda \Psi_{,x}] \quad (\text{A11})$$

$$\delta T_t^R = -\frac{1}{4\pi} \Psi_{,R} \bar{\psi}_{,t} \quad (\text{A12})$$

$$Q^2 \delta G_t^R = -\eta_{,tx} - \frac{x(\eta - \lambda)_{,t}}{x^2 + 1} \quad (\text{A13})$$

$$Q^2 \delta G_t^t = \quad (\text{A14})$$

$$= \eta_{,xx} + \frac{3x\eta_{,x} + \eta}{1 + x^2} - \frac{x\lambda_{,x}}{x^2 + 1} - \frac{x^2 + 2}{(x^2 + 1)^2} \lambda$$

$$Q^2 \delta G_R^R = \frac{x\eta_{,x} + \eta}{x^2 + 1} - \eta_{,tt} - \frac{x^2 \lambda}{(x^2 + 1)^2} \quad (\text{A15})$$

$$Q^2 \delta G_\theta^\theta = \quad (\text{A16})$$

$$= \frac{\eta_{,xx} - \lambda_{,tt} - \eta_{,tt}}{2} - \frac{x(\lambda_{,x} - 2\eta_{,x})}{2(x^2 + 1)} - \frac{\lambda}{(x^2 + 1)^2}$$

From the condition $\delta G_t^t = -\delta G_R^R$ we get

$$\eta_{,tt} = \eta_{,xx} + \frac{x(4\eta - \lambda)_{,x}}{1 + x^2} + 2\frac{\eta - \lambda}{1 + x^2} \quad (\text{A17})$$

and the condition $\delta G_t^t = \delta G_\theta^\theta$ leads to equation

$$\lambda_{,tt} + \eta_{,tt} = \quad (\text{A18})$$

$$= -\left[\eta_{,xx} + \frac{x(4\eta - \lambda)_{,x}}{1 + x^2} + 2\frac{\eta - \lambda}{1 + x^2} \right] = -\eta_{,tt}$$

and using one of the solution (A18) we get

$$\lambda = -2\eta, \quad \eta_{,tt} = \eta_{,xx} + \frac{6x}{1 + x^2} \eta_{,x} + \frac{6\eta}{1 + x^2} \quad (\text{A19})$$

For $\eta \propto \exp(i\omega t)$ we get

$$\eta_{,xx} + \frac{6x}{x^2 + 1} \eta_{,x} + \left[\omega^2 + \frac{6}{x^2 + 1} \right] \eta = 0 \quad (\text{A20})$$

$$\eta = -\frac{\lambda}{2} = \frac{f}{(1 + x^2)^{3/2}}, \quad \bar{\psi} = \frac{f_{,x}}{2\sqrt{1 + x^2}}. \quad (\text{A21})$$

$$f_{,xx} + \left[\omega^2 + \frac{3}{(1 + x^2)^2} \right] f = 0 \quad (\text{A22})$$

As is seen from equations (A20)-(A22) functions $\eta(x)$, $f(x)$ and $\bar{\psi}(x)$ include even and odd modes. This means that for the even mode the function $\bar{\psi}(x)$ is extreme at $x = 0$ while for the odd mode $\bar{\psi}(0) = 0$.

The Eq. (A22) is identical to the one-dimensional Schrödinger's equation with the energy $E = \omega^2$ and the potential box

$$U = \frac{-3}{(1 + x^2)^2}, \quad I = \int_{-\infty}^{\infty} U dx = -1.5\pi \approx -4.7 \quad (\text{A23})$$

For such equations the general structure of solution is well known. For example, for $\omega^2 > 0$ we have,

$$f = e^{i\omega t} \begin{cases} c_1 e^{i\omega x} & \text{for } x \ll -1 \\ c_2 e^{-i\omega x} + c_3 e^{i\omega x} & \text{for } x \gg 1 \end{cases} \quad (\text{A24})$$

what are the incident and reflected waves for $x \gg 1$ and the traveled wave for $x \ll -1$.

For the potential (A23) there is at least one solution with $\omega^2 = -\alpha^2$, $3 \geq \alpha^2 \geq 0$ and for finite $f(x)$ at $x \rightarrow \pm\infty$ we expect to get

$$f \sim e^{\alpha t} \begin{cases} c_1 \exp(\alpha x) & \text{for } x \ll -1 \\ c_2 \sin(\sqrt{3 - \alpha^2} x + \phi) & \text{for } -1 \ll x \ll 1 \\ c_3 \exp(-\alpha x) & \text{for } x \gg 1 \end{cases} \quad (\text{A25})$$

with $\phi = \text{const}$ and a discrete set of $\alpha = \alpha_i$, $i = 1, 2, \dots$ and $\sqrt{3} \geq \alpha_i \geq 0$.

To estimate the expected principal values α we can use solutions of the Schrödinger's equation with two similar potential boxes, namely,

$$U = U_1 = \begin{cases} 0 & \text{for } x \leq -1 \\ -3 & \text{for } -1 \leq x \leq 1 \\ 0 & \text{for } x \geq 1 \end{cases} \quad (\text{A26})$$

and

$$U = U_2 = -3\text{ch}^{-2}x, \quad -\infty \leq x \leq \infty. \quad (\text{A27})$$

For both potentials (A26 & A27) we have $I = \int_{-\infty}^{\infty} U dx = -6$ what is close to (A23). For the potential (A26) the principal value is $\alpha = \alpha_1 \approx 1.4$ and for the potential (A27) it is $\alpha = \alpha_2 = (\sqrt{13} - 1)/2 \approx 1.3$. These results indicate that for Eq. (A22) we can expect to get also the principal value $\alpha \approx 1.3 - 1.4$. In the case the growing solution of Eq. (A22) is located at $|x| \leq x_b \sim 0.6$.

This means that there is an instability in the central region of the wormhole ($|R| \leq x_b Q$) while at $|R| \geq x_b Q$ the corresponding function f is exponentially small. These results demonstrate the instability of the MT solution in respect to the small perturbations of the field Ψ .

APPENDIX B: DESCRIPTION OF THE CODE

In this appendix we describe our numerical code used to obtain the results presented in the paper.

To numerically integrate the evolution equations, eqs. (9)-(12), throughout some computational domain, we have created a new numerical code. The code has adaptive mesh refinement (AMR) capabilities in both in- and outgoing directions (contrary to our previous code, which had only AMR along the ingoing u -direction [42]). The numerical scheme used to evolve the computational cells is identical to the one we used in [42] (which evolves the unknown variables with second order accuracy), while the AMR algorithm in our code is largely based upon the design of the algorithm presented in [43].

A key feature of this AMR algorithm is the usage of a self-shadow hierarchy to compute the truncation error estimates which determines whether or not a computational cell should be split to a higher level. A self-shadow hierarchy works by always simultaneously evolving two grids, one twice the resolution of the other. Wherever the computational points in two grids coincide, it is possible to estimate the local truncation error (TE) by calculating the difference between the two solutions. This TE estimate is then measured against some limit TE_{max} and if the local truncation error is larger than that allowed, the computational cell is split to a higher level.

However, the point-wise computation of the TE as described above is, in general, not the optimal way of computing the TE as the solutions to the wave-like finite-difference equations is in general oscillatory in nature and will tend to go to zero at certain points within the computational domain [43]. Therefore, in practice, when calculating the TE we average the point-wise TE over several cells to the past of that point. Where the splitting structure allow it, we use a total of nine points to the causal past of a point to calculate the TE which determines whether a cell should be split or not. Furthermore, we calculate the TE for all four dynamic variables, if any one of these become greater than our accepted limit TE_{max} , the cell is split to a higher level.

When splitting a cell to a finer level, we need to perform an interpolation of the cells on the more coarse level in order to obtain initial data at the refinement bound-

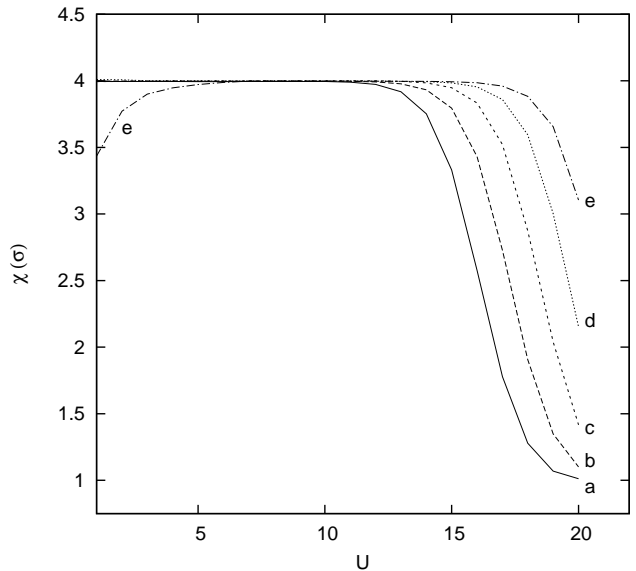


FIG. 9: Convergence factors for $\chi(\sigma)$ (see text for definition) along lines of constant u for non-AMR simulations with varying resolutions. Lines are for resolutions a) $N = 5$, b) $N = 10$, c) $N = 20$, d) $N = 40$ and e) $N = 80$.

aries for evolving the cells on the finer level (see [42] and [43] for details). If a sufficient number of adjacent points are available on the coarse parent level we use cubic Lagrange interpolating polynomials, otherwise we use linear interpolation.

APPENDIX C: TESTING OF THE CODE

We have tested our code and found it to be stable and second-order accurate. As a first, most basic test, we have verified that the code does indeed converge to the static MT solution.

In this section, we demonstrate that the code is also converging and second order accurate when including the effects of a non-trivial infalling Φ field. The initial conditions for the tests in this section are similar to those described in section V, i.e. the initial conditions for the Ψ field is set to the exact MT solution, and the form of the infalling Φ field is modeled after eq. (19) with $v_1 = 9, v_2 = 11$ and $A^\Phi = 0.0035$. The strength of this Φ pulse is such that the wormhole begins its contraction and almost, but not quite, forms a black hole within our computational domain. Our computational domain is, as for all simulations in this paper, in the range $v = [8, 28]$ and $u = [0, 20]$.

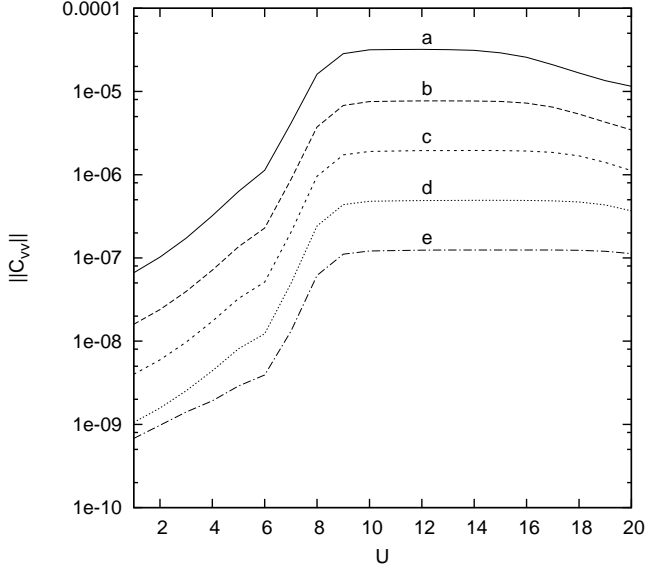
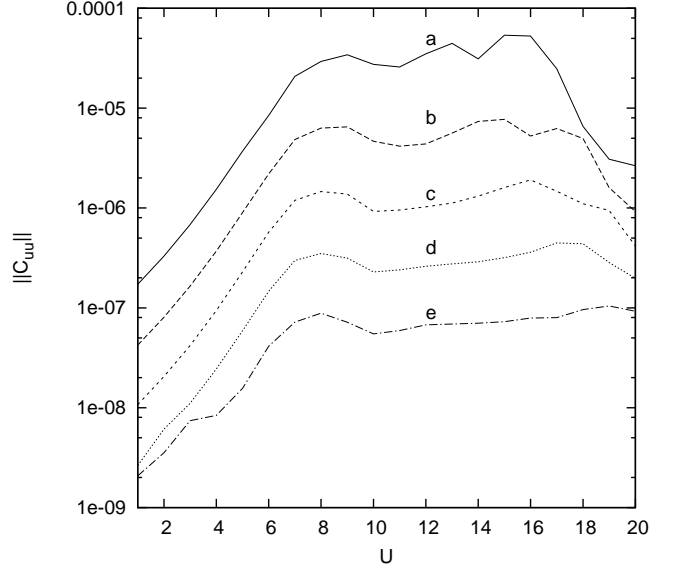
(a) $\|C_{vv}\|$ (see text) for the constraint equation (7).(b) $\|C_{uu}\|$ (see text) for the constraint equation (8).

FIG. 10: Average values of the constraint equations along lines of constant v for non-AMR simulations with varying resolutions. Lines correspond to resolutions of a) $N = 5$, b) $N = 10$, c) $N = 20$, d) $N = 40$ and e) $N = 80$.

1. Non-AMR convergence tests

First we test the basic convergence properties of the code by performing a series of simulations with varying base resolutions but with the AMR algorithm disabled.

To estimate the convergence rate along a ray of $u = \text{constant}$ we calculate :

$$\chi(x_N) \equiv \frac{1}{n} \frac{\sum_i^n |x_N^i - x_{2N}^i|}{\sum_i^n |x_{2N}^i - x_{4N}^i|} \quad (\text{C1})$$

where x_N^i denotes the dynamic variable x at the i -th grid point (along the given $u = \text{constant}$ line) of simulation with base resolution N (where $N \equiv 1/\Delta u = 1/\Delta x$). The n points over which we sum, is taken along an outgoing ray of $u = \text{constant}$. Naturally, in order for this expression to make sense, we are only using those n points which all three resolutions have in common. If the code is second-order convergent, we would expect to see a convergence factor of $\chi = 4$.

We calculate expression (C1) along lines of many $u = \text{constant}$ and for many base resolutions N , such that we get a picture of the convergence properties throughout all of our computational domain for varying resolutions. Figure 9 shows the convergence rates for the dynamical variable σ for a large number of resolutions. The figure shows that for small u (below $u = 10$), the code is second order converging. The line e (corresponding to the base resolution $N = 80$) gives the impression of less than second order convergence at small u , however a closer investigation of this behavior has revealed that this is due

to the solution in this region has reached such a high degree of convergence, that numerical roundoff errors in the numerical representation affects the calculation of the convergence factor.

For high u we see that especially for low resolutions, the convergence is far from second order. This is mainly due to the solution in this region, for low resolutions, is diverging too much from the physical solution. Such behavior is not uncommon for non-linear systems. However, as is seen, for increasing resolutions, even for high u , the convergence factor of the solution becomes increasingly higher, indicating that the numerical solution is converging with second order accuracy. That a fixed resolution can be more than adequate in one part of the computational domain and insufficient in another part is a strong indication of the need for AMR in our code. The properties of the code with AMR enables is investigated in the next subsection.

The other three dynamical variables has convergence properties similar to those of σ .

It is not enough, however, to demonstrate that our code is converging, we also need to demonstrate that it is converging to a physical solution. We do this by calculating the average value

$$\|x_N\| = \frac{1}{n} \sum_i^n x_N^i \quad (\text{C2})$$

of the two constraint equations along lines of constant u for different resolutions. Figure 10 demonstrates that both of the constraint equations is converging towards zero for increasing resolution, thus demonstrating that

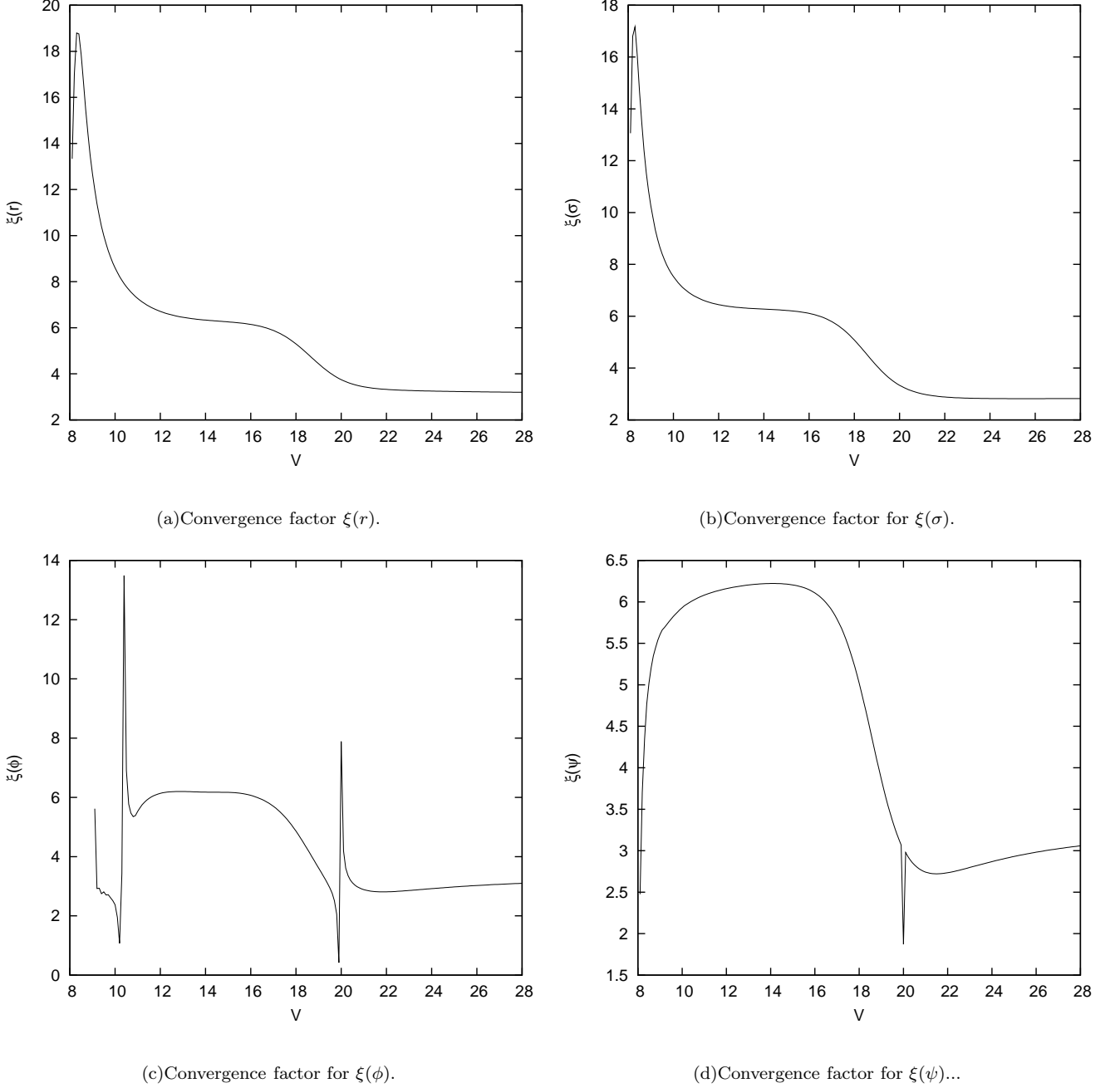


FIG. 11: Convergence factors $\xi(x)$ along line of $u = 20$ for simulations with AMR enabled, see text for details.

our numerical solution is indeed converging towards a physical solution for increased resolutions.

2. AMR convergence tests

An important component of our code is its AMR capabilities. In this subsection we demonstrate the behavior of the code when the AMR algorithm is enabled.

For each simulation we must specify a resolution of the underlying base grid and we must also specify a trunca-

tion error acceptance limit (TE_{max}) which determines whether or not a computation cell should be split. For all simulations we allow for a maximum of 8 splitting levels (including the basegrid and it's self-shadowing level, see Appendix B and [43]).

To investigate the convergence properties for the code with AMR enabled we did three simulations with the same initial conditions as in the previous section, varying the base resolution and TE_{max} to mimic the doubling of resolution from one simulation to the next as follows; The lowest resolution simulation has a base resolution

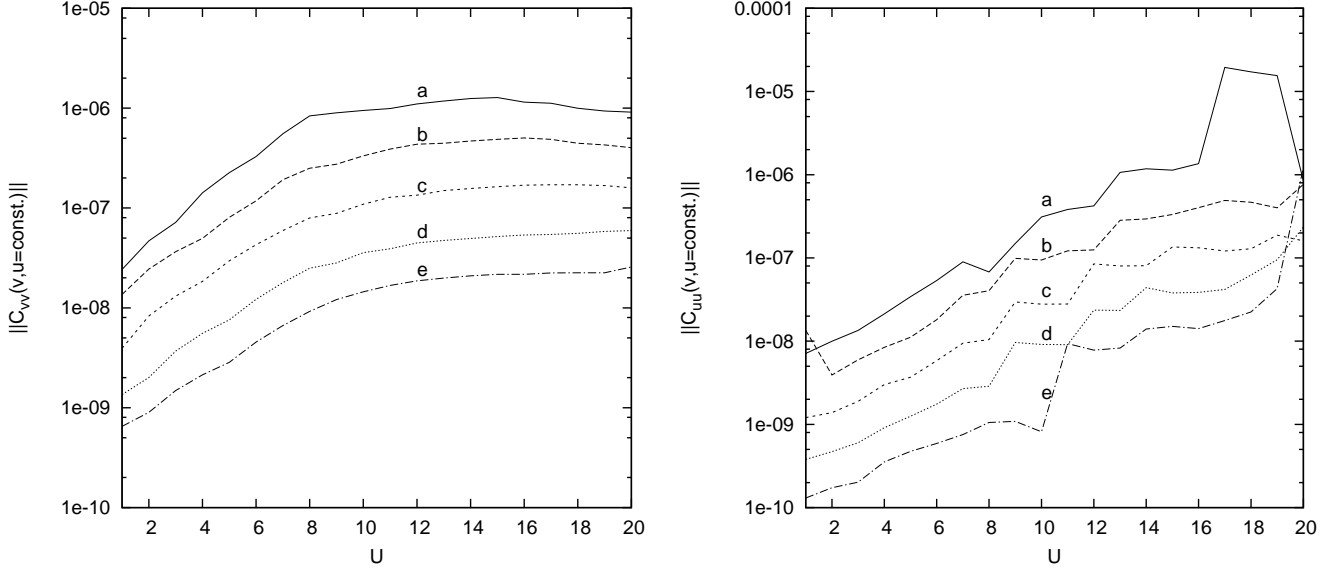
(a) Average of C_{vv} along lines of constant u for varying TE_{max} .(b) Average of C_{uu} along lines of constant u for varying TE_{max} .

FIG. 12: Average residual of the constraint equations (along lines of constant u) for fixed base resolution $N = 5$, AMR enabled and varying TE_{max} . Lines are for a) $TE_{max} = 1e - 8$, b) $TE_{max} = 1e - 9$, c) $TE_{max} = 1e - 10$, d) $TE_{max} = 1e - 11$ and e) $TE_{max} = 1e - 12$.

of $N = 5$ (corresponding to a total of 100 gridpoints) and a TE_{max} of τ_N , the second simulation has a base resolution of $N = 10$ and a TE_{max} of $\tau_{2N} = \tau_N/4$ and the last simulation has a base resolution of $N = 20$ and a TE_{max} of $\tau_{4N} = \tau_N/16$. The varying of TE_{max} mimics the doubling of the resolution from one simulation to the next (under the assumption of second order convergence) and in principle the splitting level hierarchies should be similar for all three simulations. In practice, however, we do not obtain identical splitting hierarchies because the splitting structure is determined by the TE estimate and this will not scale exactly as the leading order part of the actual truncation error, which decreases by a factor of 4 each time the mesh spacing is halved for a second order accurate scheme [43]. Nevertheless, this test gives a good indication of the behavior of the code and if the test showed non-convergent results it would indicate problems with the implementation.

In this section, to calculate the convergence factor, we use data along $u = 20$ (i.e. the upper border of our computational domain). As this is the outer border of our computational domain, convergence here is a good indication of the behavior throughout our domain. Furthermore, this border is closest to the $r = 0$ singularity (that would be formed within our domain if the initial Φ -pulse had been slightly stronger), hence it is the most demanding place in our domain to test convergence.

Figure 11 shows the convergence factors :

$$\xi(x_N) \equiv \frac{1}{n} \frac{|x_N^i - x_{2N}^i|}{|x_{2N}^i - x_{4N}^i|} \quad (C3)$$

where x_N^i denotes the dynamic variable x at the i -th grid point (along $u = 20$) of simulation with base resolution N (where $N \equiv 1/\Delta u = 1/\Delta u$). Naturally, in order for this expression to make sense, we are only using those n points which all three resolutions have in common. If the code is second-order convergent, we would expect to see a convergence factor of $\xi = 4$.

In the figure, we see that for small v , in general, the factors are greater than 4, whereas for larger v , in general the convergence factors are smaller. It is clear that, especially for r and σ we are seeing unrealistically high convergence factors for early v , however, as noted above, it would be surprising to recover perfect second order convergence in this test. The important point of fig. 11, however, is that all the dynamic variables are converging, also when the AMR algorithm is turned on, and that this convergence is in good agreement with that expected.

Finally we show the results of a series of simulations with a fixed base resolution of $N = 5$ but varying the TE_{max} limits. In fig. 12 is shown the average of each of the constraint equations along lines of constant u , i.e. :

$$||x_{TE}|| = \frac{1}{n} \sum_i^n |x_{TE}^i| \quad (C4)$$

(where x is the constraint equation, i denotes the i -th out of n point along a line of constant u).

We see that both the constraint equations are converging towards zero (although, for this test it makes no sense to talk about the "convergence rate"). It is also visible

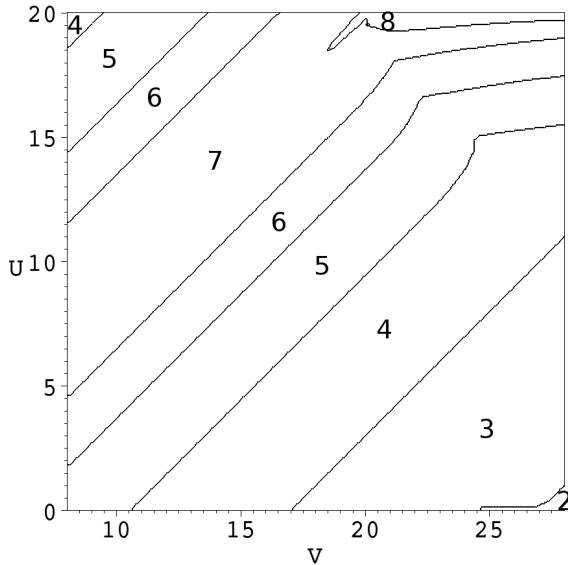


FIG. 13: Demonstration of splitting level structure. Numbers indicate the number of split levels including the base shadow hierarchy (i.e. “2” is most basic structure possible in our configuration).

from fig. 12, that for increasing u , the constraint equations are increasingly violated. This is only expected and is caused partly by truncation errors being accumulated

throughout the computational domain and partly by the solution getting close to the region where a black hole is formed and hence also the region where strong gradients in the dynamic variables puts larger requirements on the numerical code.

Another important point related with an AMR algorithm is how smooth the splitting levels are. It is well known that when a higher splitting level is introduced, this usually introduce high frequency noise into the solution near the borders of higher resolution [43], which, in the worst case, may ultimately lead to the crash of the code. Because of this, it is highly desirable to have as few splitting levels as possible and to have as few splitting borders as possible. Figure 13 illustrates the splitting level structure for a typical simulation with base resolution and TE_{max} equal to the settings used in our actual simulations in sections V-VII.

It is seen that the splitting structures are very smooth with no sudden or sporadic jumps, indicating that our implementation of the AMR algorithm is very successful. In [43] it is suggested to use a small numerical dissipation, but we have found that this is not necessary with our code, for the simulations presented in this paper.

Finally it should be noted, that for all results presented in this paper, we have performed a great number of simulations with varying base resolutions and TE_{max} . This we did to be certain that the results on which we have based our conclusions, had converged to such a degree that we could trust the results to be representative of the underlying physics.

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